Intro Video: Section 5.5
Integration by Substitution (part 1)

Math F251X: Calculus I

What is substitution?

Know: $\int f(x) \pm g(x)\, dx = \int f(x)\,dx \pm \int g(x)\,dx$

since $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

Substitution: reverses the chain rule

Right now... in order to integrate $\int f(x)\,dx$, we need to recognize that $f(x) = F'(x)$ for some $F(x)$.

$$\int x^3\,dx = \frac{x^4}{4} + C \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = \frac{\cancel{4}x^3}{\cancel{4}}$$

Idea behind substitution: replace stuff inside the integral in a clever way so that we get something that is easier to integrate!

The Substitution Rule:

If $u = g(x)$, then $\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$.

Example: $\boxed{\int \sqrt{x^3+1}\ \boxed{3x^2 dx}}$

Let $u = x^3+1$. $\Rightarrow \frac{du}{dx} = 3x^2$

$\Rightarrow du = 3x^2 dx$

$= \int \sqrt{u}\ du$

$= \int u^{1/2}\, du = \frac{2}{3} u^{3/2} + C = \boxed{\frac{2}{3}(x^3+1)^{3/2} + C}$

Example: Evaluate $\int x \sin(x^2)\,dx$

Let $u = x^2$. Then $\frac{du}{dx} = 2x \Rightarrow \frac{du}{2x} = dx$.

$$\int x \sin(x^2)\,dx = \int \cancel{x} \sin(u) \frac{du}{2\cancel{x}}$$

$$= \int \frac{\sin(u)}{2}\,du = \frac{1}{2}\int \sin(u)\,du$$

$$= \frac{1}{2}(-\cos(u)) + C = \boxed{-\frac{1}{2}\cos(x^2) + C}$$

Example: $\int (1-2x)^9 \, dx$

Let $u = 1-2x \Rightarrow \frac{du}{dx} = -2 \Rightarrow \frac{du}{-2} = dx$

$$\text{So } \int (1-2x)^9 \, dx = \int u^9 \cdot \frac{du}{-2} = -\frac{1}{2}\int u^9 \, du$$

$$= -\frac{1}{2}\,\frac{u^{10}}{10} + c = -\frac{1}{2}\left(\frac{(1-2x)^{10}}{10}\right) + C$$

$$= \frac{-1}{20}(1-2x)^{10} + c$$

$$\frac{d}{dx}\left(\frac{-1}{20}(1-2x)^{10} + c\right) = +\frac{1}{\cancel{20}}\left(\cancel{10}(1-2x)^9(+\cancel{2})\right)$$

$$= (1-2x)^9$$

Example: $\int \boxed{\sin(x) \cdot \sin(\cos(x))}\, dx$

Try $u = \sin(x)$. then $\frac{du}{dx} = \cos(x) \Rightarrow \frac{du}{\cos(x)} = dx$.

$$\int \sin(x) \cdot \sin(\cos(x))\,dx = \int u \cdot \sin(\cos(x)) \cdot \frac{du}{\cos(x)}$$

Try $u = \sin(\cos(x))$. Then $\frac{du}{dx} = \cos(\cos(x))(-\sin(x)) \Rightarrow$

$$\frac{du}{-\cos(\cos(x))\sin(x)} = dx \Rightarrow \int \frac{\cancel{\sin(x)} \cdot \sin(\cos(x))}{-\cos(\cos(x))\,\cancel{\sin(x)}}\, du$$

Try $u = \cos(x)$. Then $\frac{du}{dx} = -\sin(x) \Rightarrow \frac{du}{-\sin(x)} = dx$

$$\Rightarrow \int \sin(x)\sin(\cos(x))\,dx = \int \cancel{\sin(x)} \sin(u) \cdot \frac{du}{-\cancel{\sin(x)}} = -\int \sin(u)\,du$$

$$= -(-\cos(u)) + C = \cos(\cos(x)) + C$$

$$\frac{d}{dx}(\cos(\cos(x))) = -\sin(\cos(x))(-\sin(x)) = \sin(x)\sin(\cos(x)) \checkmark$$

Substitution and definite integrals

$$\int_a^b f(g(x))\, g'(x)\, dx$$

Method #1: Use substitution to compute the antiderivative / indefinite integral and then use FTC 2 once you've found an antiderivative

Method #2: Use the fact that

$$\int_{x=a}^{x=b} f(g(x))\, g'(x)\, dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du$$

$\longrightarrow$ This is often easier!

Example: evaluate $\int_0^{\pi/2} (\cos(x))^3 \sin(x)\,dx$

Let $u = \cos(x) \Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow \frac{du}{-\sin(x)} = dx$

If $x = 0$, $u = \cos(0) = 1$ and $x = \pi/2 \Rightarrow u = \cos(\pi/2) = 0$.

$$\text{So } \int_{x=0}^{x=\pi/2} (\cos(x))^3 \sin(x)\,dx = \int_{u=1}^{u=0} u^3 \sin(x) \cdot \frac{du}{-\sin(x)}$$

$$= -\int_{u=1}^{u=0} u^3\,du = \int_0^1 u^3\,du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}.$$

Method #1: $\int (\cos(x))^3 \sin(x)\,dx = \int u^3 \sin(x)\left(\frac{du}{-\sin(x)}\right) = -\int u^3\,du$

$= -\frac{u^4}{4} + C \Rightarrow -\frac{(\cos(x))^4}{4} + C$. So $\int_0^{\pi/2} (\cos(x))^3 \sin(x)\,dx$

$$= -\frac{(\cos(x))^4}{4}\Big|_0^{\pi/2} = -\frac{(\cos(\pi/2))^4}{4} - \left(\frac{-(\cos(0))^4}{4}\right) = 0 + \frac{1}{4} = \frac{1}{4}$$